First Order/Linear Functions
(really, Affine)

$$A : x \to sx + a.$$

Two points of view:

i) function

ii) **transformation**

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Two algebra structures:

$$A(x) = sx + a, \quad B(x) = tx + b.$$

1: **Addition:**
$(A + B)(x) = (s + t)x + (a + b).$

2: **Composition:**

$$\begin{aligned} (A \circ B)(x) &= A(B(x)) \\ &= A(tx + b) = s(tx + b) + a \\ &= stx + (a + sb). \end{aligned}$$

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Inverse:

$$A_{s,a}(x) = sx + b$$

$$(A_{s,a})^{-1} = A_{\frac{1}{s},-\frac{a}{s}}.$$

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Application

Solve $\quad sx + a = y.$

This says $\quad A_{s,a}(x) = y.$
Hence

$$x = (A_{s,a})^{-1}(y)$$

$$= A_{\frac{1}{s},-\frac{a}{s}}(y) = \frac{y}{s} - \frac{a}{s}.$$

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Translations, Dilations

Translation by a: $T_a : x \to x + a$.

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Dilation by s: $D_s : x \to sx$.

Then $A_{s,a} = T_a \circ D_s$.

(Every affine map is uniquely composed from a dilation and a translation.)

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Affine Recursion

$$x_{n+1} = ax_n + b = A_{s,b}(x_n).$$

Special cases:

$$x_{n+1} = x_n + b = T_b(x_n).$$

$$x_{n+1} = sx_n = D_s(x_n).$$

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Examples:

1. Sue has \$1,000 in her savings account.
Every month she adds \$50.
How much will she have in her account
after 3 months? 6 months? 2 years?

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2. Sue has \$1,000 in her savings account.

Every month, the bank pays interest of .1%

How much will be in her account

after 3 months? 6 months? 2 years? .

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3. Sue has \$1,000 in her savings account.
Every month, the bank pays interest of .1%,
and she adds \$50.
How much will be in her account
after 3 months? 6 months? 2 years?

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4. Sue has a credit card, with a balance of \$1,000.
Every month, the credit card company
charges interest of 1%,
and Sue makes the minimum \$15 payment.
When will Sue's card balance be \$0?
How much will she pay in all?

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5. Stu takes 500mg of leprosol every 12 hours.
Between doses, his liver breaks down 70%
of the drug.
How much leprosal is in Stu's system
after he takes his 3rd dose? 6th dose?10th?20th?

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6. What is the sum $S_n = \sum_{j=0}^{n} br^j$?

Note that

$$S_{n+1} = \sum_{j=0}^{n+1} br^j \qquad = b + \sum_{j=1}^{n+1} br^j$$

$$= b + \sum_{j=0}^{n} br^{j+1} = b + r\left(\sum_{j=0}^{n} ar^j\right)$$

$$= b + rS_n.$$

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Solving Affine Recursions

Framework:

$$x_{n+1} = A_{s,b}(x_n) \to x_n = (A_{s,b})^{(n)}(x_o).$$

(n-fold composition)

How to compute $(A_{s,b})^{(n)}$?

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Easy cases

Translations: $T_b(x) = x + b$

$$\Rightarrow (T_b)^n(x) = x + nb = T_{nb}(x).$$

Dilations: $D_s(x) = sx$

$$\Rightarrow (D_s)^n(x) = s^n x = D_{s^n}(x).$$

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Fixed Points and Conjugation

For $s \neq 1$, D_s has 0 as its unique fixed point:

$$D_s(x) = x \iff x = 0.$$

We can check:

$$T_c \circ D_s \circ T_{-c} = A_{s,(1-s)c}.$$

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That is: when $s \neq 1$,
every $A_{s,b}$ is conjugate to D_s,
by the translation T_c, with $c = \frac{b}{1-s}$.
Also, $A_{s,b}$ has c as its unique fixed point.
Hence

$$(A_{s,b})^{(n)} = (T_c D_s T_c^{-1})^{(n)} = T_c (D_s)^{(n)} T_c^{-1}$$

$$T_{c(1-s^n)} D_{s^n} = A_{s^n, b\frac{1-s^n}{1-s}}.$$

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Action on Functions

Precomposition and postcomposition:

$$f \to A_{t,d} \circ f \circ A_{s,b}.$$

$$A_{t,d} \circ f \circ A_{s,b}(x) = tf(sx+b) + d.$$

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Solving Quadratics

Given

$$Q(x) = \alpha x^2 + \beta x + \gamma = 0.$$

Then

$$Q(x) = A_{s,a} \circ x^2 \circ T_b,$$

where

$s = \alpha,$

$a = \beta/2s = \frac{\beta}{2\alpha},$

$\underline{=}\gamma - sa^2 = \frac{4\alpha\gamma-\beta^2}{4\alpha}.$

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Analog for cubics:

$$\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$$
$$= T_{s,a} \circ (x^3 + \epsilon x) \circ T_{t,b},$$

where

$$\epsilon = sgn(3\alpha_1\alpha_3 - \alpha_2^2) = 0, \pm 1$$

Set $\delta = |3\alpha_1\alpha_3 - \alpha_2^2|$. Then

$t = \alpha_3\sqrt{\frac{3}{\delta}}, \qquad b = \frac{\alpha_2}{3\alpha_3}t = \alpha_2\sqrt{\frac{1}{3\delta}},$

$s = \frac{\alpha_3}{t^3} = \frac{\sqrt{3}\delta^{\frac{3}{2}}}{9\alpha_3^2}, \qquad a = \alpha_o + \frac{2\alpha_2^3 - 9\alpha_1\alpha_2\alpha_3}{27\alpha_3^2}.$

(Not the usual approach.) .

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Symmetry characterization of standard functions

Theorem: If a function $F : x \to y$ (or its graph)
is invariant by a non-discrete subgroup
of affine pre- and post- compositions,
then up to affine transformations, F is

a) linear;
b) a power function;
c) an exponential function; or
d) a logarithmic function

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Link to Geometry

The complex $ax + b$ group
is the group of

orientation- preserving

similarities
of the Euclidean/complex plane.

(Euclidean distance = complex absolute value.)